COWLES COMMISSION DISCUSSION PAPER: ECONOMICS: NO. 264A LPC 411A

ALLOCATION OF RESOURCES IN PRODUCTION

II APPLICATION TO TRANSPORTATION

By

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This is the second part of the notes of lectures given in the Spring of 1949 by the former author and recorded by the latter. The material can be regarded as an elaboration of the contents of LPC 401.

(Errata have been given effect to.)

Lecture 11 -- May 19, 1949.

APPLICATION OF LINEAR ALLOCATION MODELS TO TRANSPORTATION

A Static Model with two ports only. We shall begin with a consideration of a simplified shipping model, keeping in mind that it is possible to translate ships into trains or aircraft, and ports into terminals or airfields. Our present example involves two ports, A and B. The model is given in Table 17 below (see p. 94).

Table 17 is to be read in the same way our tables of technological coefficients for previous models were read. There are two activities for each port, loading and discharging cargo; there are two activities whereby a ship is moved from one port to the other, sailing with cargo and sailing in ballast (without cargo). It should be noted that the Greek letters appearing in the last three rows of the table symbolize numbers which have the dimension "time" measured in months. For example, the number λ_A denotes the number of months required to load a ship in Port A. Since the activities are all in units of "ships per month", when λ_A is multiplied by $\bar{x}_A$, a number having the dimension "ships" results. That is to say,

$$(11.1) \quad \left(\bar{x}_A \text{ ships/month}\right)\left(\lambda_A \text{ months}\right) = (\bar{x}_A)\ (\lambda_A) \text{ ships}$$

tied up at any instant (or on the average for many instants of time) in loading at A. This is also the dimension of z, the total fleet in use. Similarly with the other Greek letters.

Note also that we are assuming that all the ships are of the same type, and are therefore completely interchangeable.

Tab 4-19

Commodities	Units	Symbol	Port A: Loading $\bar{X}_A$	Port A: Discharging X_A	Port B: Loading $\bar{X}_B$	Port B: Discharging X_B	Route A to B: Sailing Loaded $\bar{X}_{AB}$	Route A to B: Sailing Empty X_{AB}	Route B to A: Sailing Loaded $\bar{X}_{BA}$	Route B to A: Sailing Empty X_{BA}
Final:										
Cargo from A to B	Shiploads per mo.	y_{AB}				1				
Cargo from B to A	"	y_{BA}		1						
Intermediate:	Ships per mo.									
Net flow of loaded ships										
at A for B		0	1				-1			
at A from B	"	0		-1					1	
at B for A	"	0			1				-1	
at B from A	"	0				-1	1			
Net flow of empty ships										
at A	"	0	-1	1				-1		1
at B	"	0			-1	1		1		-1
Primary:										
Shipping	Ships	Z	$-\lambda_A$	$-\delta_A$	$-\lambda_B$	$-\delta_B$	$-\bar{\sigma}_{AB}$	$-\sigma_{AB}$	$-\bar{\sigma}_{BA}$	$-\sigma_{BA}$
Use of Port A	Berths	Z_A	$-\lambda_A$	$-\delta_A$						
Use of Port B	"	Z_B			$-\lambda_B$	$-\delta_B$				

Activities (In Units of Ships per Month)

An examination of Table 17 reveals that the requirement that the net flows of intermediate commodities in the category "net flows of loaded ships" be zero implies that certain activities must be operated in the same amount. For example, the requirements that the net flow of loaded ships at A for B be zero, together with the requirement that the net flow of loaded ~~xx~~ ships at B from A be zero imply

$$(11.2a) \qquad \bar{\bar{x}}_A = \bar{\bar{x}}_{AB} = x_B .$$

Similarly, the requirement that the remaining two net flows in the category be zero implies

$$(11.2b) \qquad \bar{\bar{x}}_B = \bar{\bar{x}}_{BA} = x_A$$

But if three activities are required to be carried out in equal amounts, we can combine the three separate activities into a single activity, eliminating the intermediate products between them. Thus, we can define a new activity, "transporting cargo from A to B" which consists of loading at A, sailing loaded from A to B and discharging cargo at B. Similarly we can define a new activity, "transporting cargo from B to A. In terms of these new activities our matrix of technical coefficients appears as in Table 18. (See p. 96).

Notice that

$$(11.3a) \qquad \tau_{AB} = \lambda_A + \bar{\bar{\sigma}}_{AB} + \delta_B$$

that is, the number of months a ship is tied up in transporting cargo from A to B is the number of months required to load it at A, to make the voyage from A to B and to discharge it at B. Similarly,

$$(11.3b) \qquad \tau_{BA} = \lambda_B + \bar{\bar{\sigma}}_{BA} + \delta_A$$

The number λ_A of months a berth is tied up in Port A by the activity

Table 18

Commodities	Unit	Symbol	Activities (In Units of Ships per Month): Route A to B — Transporting Cargo $\bar{X}_{AB}$	Route A to B — Sailing Empty X_{AB}	Route B to A — Transporting Cargo $\bar{X}_{BA}$	Route B to A — Sailing Empty X_{BA}
Final:						
Cargo from A to B	Ship-loads per mo	y_{AB}	1			
Cargo from B to A	"	y_{BA}			1	
Intermediate: net flows of empty ships at A	Ships per mo.	0	−1	−1	1	1
at B	"	0	1	1	−1	−1
Primary:						
Shipping	Ships	z	$-\tau_{AB}$	$-\sigma_{AB}$	$-\tau_{BA}$	$-\sigma_{BA}$
Use of Port A	Berths	z_A	$-\lambda_A$		$-\delta_A$	
Use of Port B	"	z_B	$-\delta_B$		$-\lambda_A$	

"transporting cargo from A to B" is the number of months it takes to load a ship at Port A. Similarly, δ_B, is the number of months a berth is tied up in Port B.

The model as it now stands contains three factors which could conceivably be bottlenecks, depending on their given initial stocks.

These are shipping, z, facilities (berths) in Port A, z_A, and facilities in Port B, z_B. We shall simplify our model further by assuming that port facilities at both A and B are known to be so plentiful as to preclude their being a bottleneck. Thus, we are no longer interested in how many berths are tied up by any of our activities. This eliminates from consideration the last two rows of our technology matrix, leaving us with three variables of interest, y_{AB}, y_{BA} and z. Our problem is to find the efficient point set in the space of these three variables.

The equations obtained from Table 13 are

$$(11.4a) \qquad y_{AB} = \bar{x}_{AB}$$
$$(11.4b) \qquad y_{BA} = \bar{x}_{BA}$$
$$(11.4c) \qquad 0 = -\bar{x}_{AB} - x_{AB} + \bar{x}_{BA} + x_{BA}$$
$$(11.4d) \qquad 0 = \bar{x}_{AB} + x_{AB} - \bar{x}_{BA} - x_{BA}$$
$$(11.4e) \qquad z = -\tau_{AB}\bar{x}_{AB} - \sigma_{AB}x_{AB} - \tau_{BA}\bar{x}_{BA} - \sigma_{BA}x_{BA}$$

We can use (11.4a) and (11.4b) to eliminate $\bar{x}_{AB}$ and $\bar{x}_{BA}$. We also note that (11.4d) is implied by (11.4c). We can therefore omit it. (These two equations are the same because we have a closed system, without any activities that introduce or remove ships. The net flows in one port must equal the net flows in the other port). Thus, we are left with the following equations:

$$(11.5a) \qquad 0 = -y_{AB} - x_{AB} + y_{BA} + x_{BA}$$
$$(11.5b) \qquad -z = \tau_{AB}y_{AB} + \sigma_{AB}x_{AB} + \tau_{BA}y_{BA} + \sigma_{BA}x_{BA}$$

remembering that

$$(11.5c) \qquad y_{AB} \geq 0 \text{ and}$$
$$y_{BA} \geq 0 \text{ because of (11.4a) and (11.4b).}$$

(11.5d) $x_{AB} \geq 0$ and $x_{BA} \geq 0$;

and that by their very nature all performance times are positive

(11.5e) $\tau_{AB} > 0$; $\sigma_{AB} > 0$; $\tau_{BA} > 0$; $\sigma_{BA} > 0$.

Our problem [to find the efficient point set in the space (y_{AB}, y_{BA}, z)] may be stated by asking: for given y_{AB} and y_{BA}, what is the maximum z (or the minimum $-z$)?

<u>Numerical</u> <u>Example</u>. Specify

(11.6a) $y_{AB} = 25$ shiploads per month

(11.6b) $y_{BA} = 15$ shiploads per month

Assume (11.6c) $\sigma_{AB} = \sigma_{BA} = 1/2$ month

(11.6d) $\tau_{AB} = \tau_{BA} = 1$

The amount of cargo to be transported from A to B exceeds the amount to be transported from B to A. We will have to send some ships from B to A empty. However, it would obviously be wasteful to send empty ships from A to B. Therefore, efficiency requires

(11.7a) $x_{AB} = 0$

From (11.5a), (11.6a), (11.6b) and (11.7a) it follows that

(11.7b) $x_{BA} = 10$

Substituting in (11.5b) we find

(11.7c) $-z = 1.25 + (1/2)0 + 1.15 + (1/2)10 = 45$

Thus, the point $(y_{AB}, y_{BA}, z) = (25, 15, 45)$ is an efficient point. If we followed this procedure for all values of y_{AB} and y_{BA} satisfying equations (11.5a-d) we would find all the efficient points. An algebraic method would, however, be simpler.

We begin as in our numerical example with a given transportation program; i.e., with given values of y_{AB} and y_{BA}. Thus, rewriting (11.5a) we have

$$(11.8a) \quad x_{AB} - x_{BA} = y_{BA} - y_{AB} = \text{a given number.}$$

We distinguish between two cases.

<u>Case I</u>. $y_{BA} - y_{AB} \quad 0$

(11.8a) still permits us to add to or subtract from both x_{AB} and x_{BA} the same number, subject to (11.5d). Looking at (11.5b) in the light of (11.5e), we see that $-z$ is minimized by subtracting the largest number that does not violate (11.5d). Thus, efficiency requires

$$(11.8b) \quad x_{BA} = 0 \quad \text{and hence}$$

$$(11.8c) \quad x_{AB} = y_{BA} - y_{AB} \geqslant 0$$

Substituting in (11.5b)

$$(11.8d) \quad -z = \tau_{AB} y_{AB} + \sigma_{AB}(y_{BA} - y_{AB}) + \tau_{BA} y_{BA} + 0$$

or

$$(11.8e) \quad -z = (\tau_{BA} + \sigma_{AB}) y_{BA} + (\tau_{AB} - \sigma_{AB}) y_{AB}$$

Equation (11.8e) is the equation of one facet of the efficient point set.

<u>Case II</u>. $y_{BA} - y_{AB} \leqslant 0$. Now, efficiency requires

$$(11.9a) \quad x_{AB} = 0$$

$$(11.9b) \quad x_{BA} = y_{AB} - y_{BA} \geqslant 0$$

Substituting in (11.5b), we have

$$(11.9c) \quad -z = (\tau_{BA} - \sigma_{BA}) y_{BA} + (\tau_{AB} + \sigma_{BA}) y_{AB}$$

which is the equation of the second facet of the efficient point set.

If we use the values assumed in equations (11.6c) and (11.6d) — our numerical example we can draw the graph of the efficient point set. We have then

$$(11.10a) \qquad -z = 3/2\, y_{BA} + 1/2\, y_{AB}$$

$$(11.10b) \qquad -z = 1/2\, y_{BA} + 3/2\, y_{AB}$$

Let $z = -6$

Then, Fig. 50 shows the efficient and achievable point sets in the space (y_{AB}, y_{BA}) and Fig. 51 shows the efficient (and achievable) point sets in the space of (y_{AB}, y_{BA}, z).

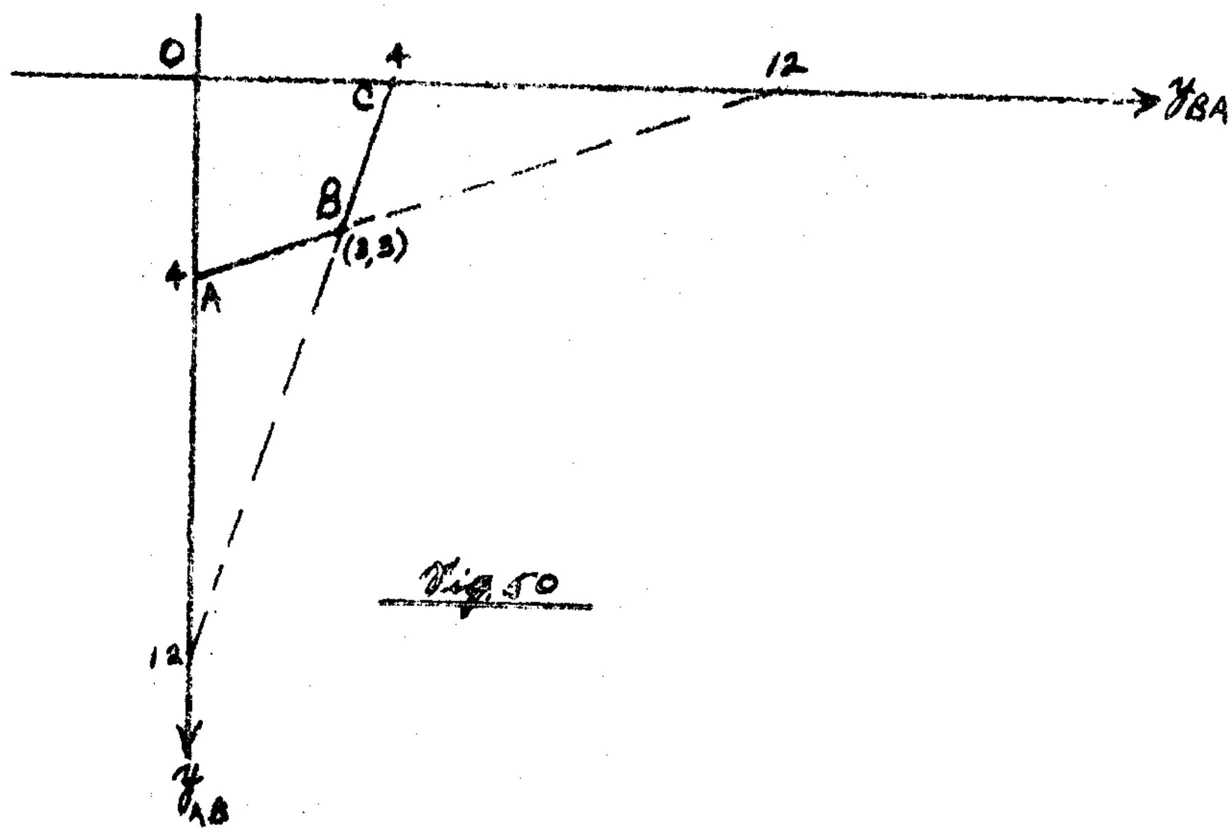

Fig. 50

The achievable point set is the area enclosed by $\overline{CABO}$; the efficient point set consists of the lines $\overline{AB}$ and $\overline{BC}$.

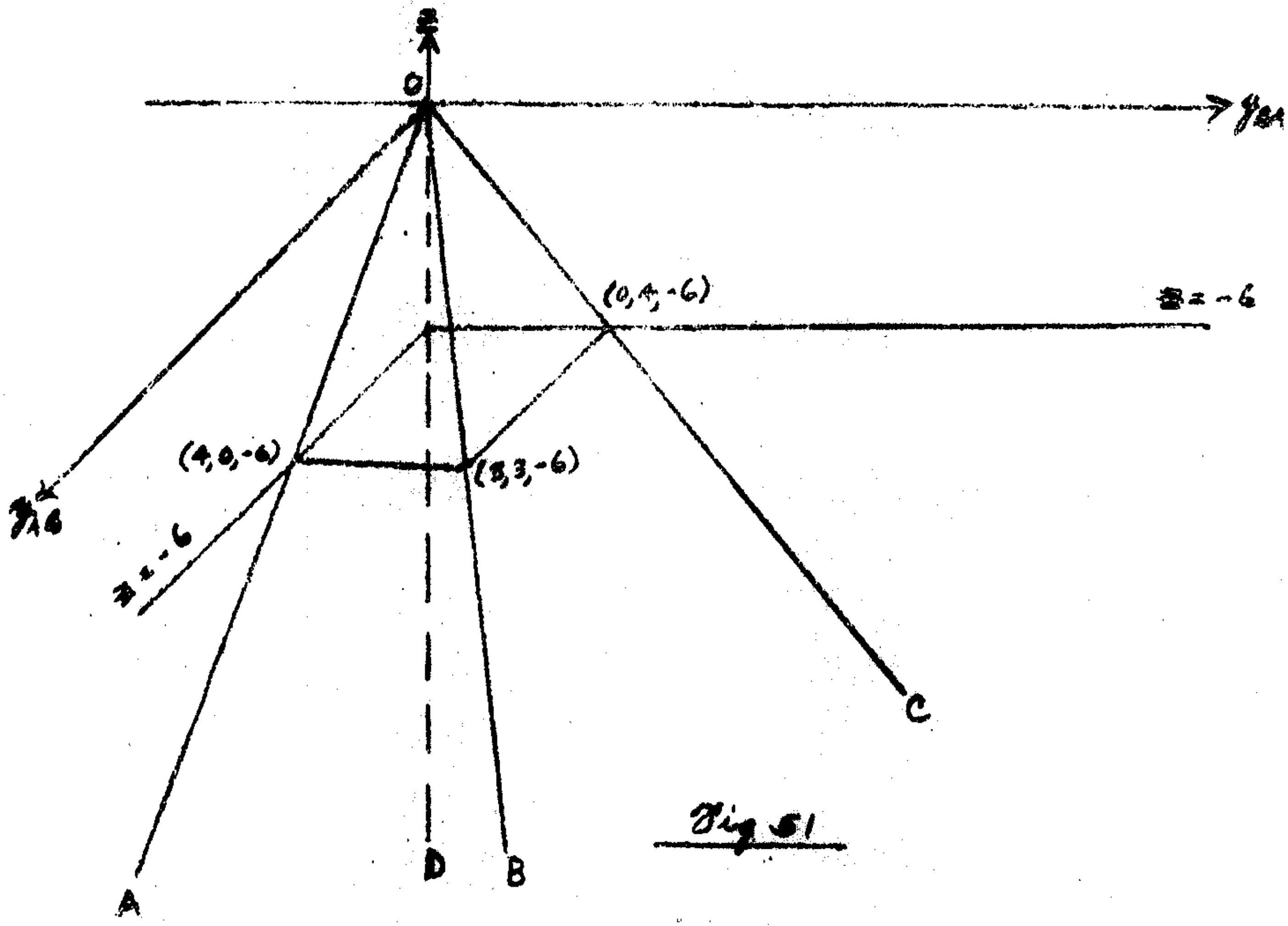

Fig 51

The achievable point set is the cone spanned on OA, OB, OC and OD. The efficient point set consists of the facets AOB and BOC. The boundary OB between the facets remains within the same vertical plane $y_{AB} = y_{BA}$ irrespective of the values of the performance times.

To increase y_{AB} with one unit on a given facet, (11.8e) or (11.9e), as the case may be, gives the required increase in shipping in use, the marginal rate of exchange of ship use for transportation performance. Since there is only one scarce primary factor in the model, shipping, we can also refer to this rate as the marginal cost of transportation in terms of shipping required. In this terminology, the marginal cost, in terms of shipping, of transporting a shipload from B to A when the dominating cargo movement is from B to A; i.e., on the facet $y_{BA} - y_{AB} \geqslant 0$, is $(\tau_{BA} + \sigma_{AB})$. This is the turn-around time for a ship carrying cargo from B to A and returning empty. On the other facet; i.e., $y_{BA} - y_{AB}$ 0, the marginal cost of transporting cargo from B to A is $(\tau_{BA} - \sigma_{BA})$ which is the cost of transferring a ship from sailing empty from B to A to sailing with cargo from B to A.

It is proper that these marginal cost coefficients be measured in months. The addition to loaded traffic from B to A is measured in ships per month, the ensuing addition to required shipping is measured in ships. The ratio is measured in

$$\text{ships/ships per month} = \text{months}$$

Lecture No. 12 -- May 24, 1949.

TRANSPORTATION MODEL (CONT'D)

In Lecture 11 we found that the equation for the two facets of our efficient point set were

(12.1a) $(y_{BA} > y_{AB})$ $\quad -z = (\tau_{BA} + \sigma_{AB})y_{BA} + (\tau_{AB} - \sigma_{AB})y_{AB}$

(12.1b) $(y_{AB} > y_{BA})$ $\quad -z = (\tau_{BA} - \sigma_{BA})y_{BA} + (\tau_{AB} + \sigma_{BA})y_{AB}$

We could have deduced these equations by a direct argument. Assume we are on the facet $(y_{BA} > y_{AB})$, say $y_{BA} = 25$, $y_{AB} = 15$. This means that there is a monthly flow of 25 loaded ships from B to A, and two flows--one of 15 loaded ships and one of 10 empty ships--from A to B. Now we send an additional ship from B to A each month. In a static model this necessarily increases the flow of empty ships from A to B from 10 to 11 a month. Thus, each month we commit one ship for $(\tau_{BA} + \sigma_{AB})$ months; i.e., the time for a loaded movement B to A followed by an empty movement A to B. Thus, at any time we have in additional employment

(12.2) (1 ship/month) $(\tau_{AB} + \sigma_{AB})$months $= (\tau_{BA} + \sigma_{AB})$ships

If we added two shiploads to monthly traffic from B to A, we would have to add $2(\tau_{BA} + \sigma_{AB})$ ships to the employed fleet, etc. Therefore the cost in ships of sending one additional shipload of cargo per month from B to A when shipments from B to A exceed those from A to B is $(\tau_{BA} + \sigma_{AB})$. This ratio applies to reductions as well as increases in shipments on the facet $y_{BA} > y_{AB}$.

If on the same facet we add a shipload per month going from A to B, this cargo is loaded on a ship that would otherwise move empty.

Therefore we simultaneously commit τ_{AB} shipments each month and save σ_{AB} shipments each month. The net effect on shipping employed at any given time is the addition of $(\tau_{AB} - \sigma_{AB})$ ships. Therefore the cost in ships of an additional shipload per month from A to B on the facet $(y_{BA} > y_{AB})$ is $(\tau_{AB} - \sigma_{AB})$. This gives equation (12.1a). By the same reasoning we could obtain (12.1b) directly. However, at $y_{BA} = y_{AB}$, note that marginal cost for an increase is not the same as marginal cost for a decrease, because marginal cost depends on which facet we move onto.

Generalization to More Ports. With only two ports in our model, we had two pairs of columns, one pair for each port. If we add another port we will obtain six pairs of columns, giving us all possible points of destination from each port for loaded and empty ships. As commodities we will have six final flows, three intermediate flows and one primary factor. We wish to find the efficient point set for this case. Our present approach will be intuitive.

We shall specify the cargo flows at each port (in Table 19) and minimize cost as measured in shipping employed.

Table 19

Required Flows	To Port A	B	C
From: Port A	0	12	15
B	8	0	18
C	25	4	0

We now strike a balance between inflows and outflows of loaded ships at each port.

Table 20

Ports	Outflow of Loaded Ships	Inflow of Loaded Ships	Net Surplus
A	27	33	6
B	26	16	-10
C	29	33	4

The column "net surplus" in Table 20 represents the monthly number of ships completing discharge at each port in excess of t number required to commence loading to maintain the prescribed cargo flow from that port. Thus, port A has 6 extra ships a month; Port B has a deficit of 10 ships a month and Port C has 4 extra ships a month. It is intuitively clear that to send empty ships from A to C or from C to A would be wasteful. Similarly, it would be wasteful to send empty ships to A or C from B. Thus, the efficient solution is to send 6 empty ships each month from A to B and 4 empty ships each month from C to B. This will absorb the net surpluses of ships in each port. Fig. 52 illustrates this.

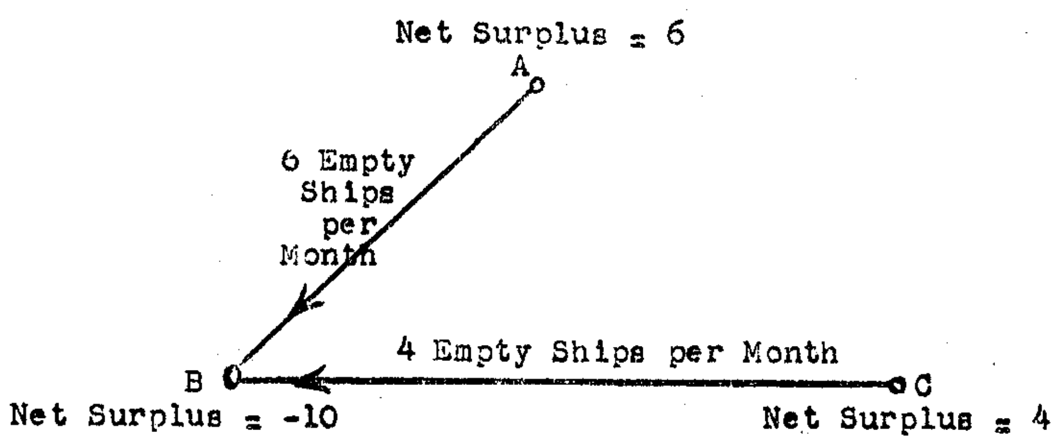

Fig. 52

Table 21. List of Areas

Number	Representative Ports	Areas
1	New York	Atlantic (incl. Gulf) Coast of Canada and U.S.
2	San Francisco	Pacific Coast of Canada and U.S.
3	St. Thomas	Mexico, Caribbean, North Coast of S. America and Brazil
4	Buenos Aires	Argentina
5	Antofagasta	West Coast of S. America
6	Rotterdam	Baltic countries, Norway, Germany, Netherlands, Belgium, Great Britain and Ireland
7	Lisbon	France, Spain, Portugal
8	Athens	Mediterranean except France and Spain
9	Odessa	Black Sea countries
10	Lagos	West Africa
11	Durban	South and East Africa
12	Bombay	Arab, Iran, India
13	Singapore	Malay, Siam, Indo China, Phillipines and Indonesia
14	Yokohama	Japan, China, Asiatic USSR
15	Sydney	Australia and New Zealand

Table 22. Net receipts of goods in overseas trade, 1925.

(1)	(2)	(3)	(4)
Area repre-sented by	All cargoes other than mineral oils		
	Received	Dispatched	Net receipts
New York	23.5	32.7	-9.2
San Francisco	7.2	9.7	-2.5
St. Thomas	10.3	11.5	-1.2
Buenos Aires	7.0	9.6	-2.6
Antofagasta	1.4	4.6	-3.2
Rotterdam*	126.4	130.5	-4.1
Lisbon*	37.5	17.0	20.5
Athens*	28.3	14.4	13.9
Odessa	0.5	4.7	-4.2
Lagos	2.0	2.4	-0.4
Durban*	2.1	4.3	-2.2
Bombay	5.0	8.9	-3.9
Singapore	3.6	6.8	-3.2
Yokohama	9.2	3.0	6.2
Sydney	2.8	6.7	-3.9
Total	266.8	266.8	0.0

Unit: Millions of metric tons.

Source: Der Guterverkehr der Weltschiffahrt, Statistisches Reichsamt, Berlin, 1928.

*The figures in columns (2) and (3) for this area contain an equal amount of traffic within the area, between smaller areas from which this area was composed.

Table 28. Distances between representative ports.

	Sydney	Yokohama	Singapore	Bombay	Durban	Lagos	Odessa	Athens	Lisbon	Rotterdam	Antofagasta	Buenos Aires	St. Thomas	San Francisco
New York								4.8	3.0			5.9	1.4	
San Francisco		4.5							7.6		4.7			
St. Thomas								5.0	3.2			4.6		
Buenos Aires					4.6	4.3		7.1	5.3					
Antofagasta		9.4						8.6	6.6					
Rotterdam									1.1					
Lisbon	10.6			5.2	6.0	2.9								
Athens	9.0		5.7	3.7	5.2	4.6	.7							
Odessa														
Lagos														
Durban	6.2		4.9											
Bombay		5.4	2.4											
Singapore		2.9												
Yokohama	4.3													

Unit: Thousands of nautical miles.

ERRATA

Page 99, lines 5 and 6 - change "parts" to "ports".

The figures in Tables 19 and 20 on p. 104 should be changed as follows:

Table 19

Required Flows		To Port	A	B	C
From:	Port A		0	12	15
	B		8	0	18
	C		25	4	0

Table 20

Ports	Outflow of Loaded Ships	Inflow of Loaded Ships	Net Surplus
A	27	33	6
B	26	16	-10
C	29	33	4

Page 105, sentence on lines 2 and 3 from below should be changed to: "This will absorb the net surpluses of ships in each port."

The tables on pages 106-108 should be numbered 21, 22, 23, successively.

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Lecture 13 -- May 23, 1949.

TRANSPORTATION MODEL (CONT'D)

Marginal Cost; Equivalence Ratios. In Lecture 12 we have solved in the 3-port case one of the problems in which we are interested; namely: the problem of finding an efficient point corresponding to a given transportation program. There remains the second important problem of finding equivalence ratios; i.e., we are not only interested in finding an efficient way to transport given amounts of cargo; we are also interested in finding the terms on which we can efficiently change these amounts. Since shipping is the sole input it is natural to mea(cost in terms of it. We shall do so.

We wish to determine the cost of an additional shipload of cargo transported each month from B to A, where the original situation is as in Figure 52; i.e., efficient transportation. Instead of explicit mathematical derivation leading to an equation for the facet of the efficient point set similar to (12.1a), we shall give an equivalent verbal argument. It is clear that sending the additional shipload of cargo per month requires the loading, sailing with cargo from B to A and discharging of a ship each month. Thus, "direct" cost; i.e., shipping engaged in loading, loaded movements, or discharging, increases by τ_{BA}. But sending an additional loaded ship to A each month leaves Port A with a net surplus of 7 ships and Port B with a net deficit of 11 ships. Therefore, an additional empty ship must be sent from A to B each month, adding σ_{AB} to the "indirect cost"; i.e., the shipping engaged in empty movements. Thus, the marginal cost of cargo tran portation from B to A—to be denoted M_{BA}—is

$$(13.1) \qquad M_{BA} = \tau_{BA} + \sigma_{AB}$$

Notice that this is the same marginal cost as in the two port case. The existence of a program involving the third port is irrelevant, as long as efficient execution of the program requires empty movements from A to B.

Although marginal cost is defined here with respect to unit increases, the concept can also be applied to decreases. We wish now to explore the range in which the marginal cost, given by equation (13.1) is applicable. We note first that equation (13.1) applies for indefinitely large increases in shipments from B to A. (We have not assumed any fixed initial stock of shipping and we have ruled out the possibility of port congestion.) In the case of decreases, however, we cannot make negative shipments from B to A, so that the most we can reduce shipping by is 8 ships per month; i.e.,

$$(13.2) \qquad \Delta y_{BA} \geq -8$$

We shall call the restriction expressed by (13.2) the feasibility limit.

Notice, however, that (13.1) ceases to apply before this limit is reached. If we decrease shipments from B to A by six ships per month, the net surplus of empty ships at A will be zero. If shipments from B to A are then further reduc by 1 ship per month, it will be necessary to send an additional empty ship from C to A and reduce the flow of empty ships from C to B by one. Thus,

$$(13.3) \qquad \Delta y_{BA} \geqq -6$$

The restriction expressed by (13.3) may be called the limit of applicabili of the marginal cost coefficient (13.1). This limit is reached at the point where some flow of empty ships "dries up". Thus, in the present case, the marginal cost given by equation (13.1) is applicable only to changes algebraica greater than -6. If $\Delta y_{BA} \leqq -6$ we move onto another facet of the efficient poin set where a different marginal cost coefficient applies. Table 24 below sho marginal costs and their limits of feasibility and applicability for the presen three-port case.

Table 24

Changes in Cargo Transportation From Port...	To Port...	Marginal Cost	Limits of Feasibility	Limits c Applicabil
B	A	$\tau_{BA} + \sigma_{AB}$	$\Delta y_{BA} \geqq -8$	$\Delta y_{BA} \geqq -6$
A	B	$\tau_{AB} - \sigma_{AB}$	$\Delta y_{AB} \geqq 12$	$\Delta y_{AB} \leqq 6$
B	C	(Compute as an exercise)		
C	B	(Compute as an exercise)		
A	C	$\tau_{AC} - \sigma_{AP} + \sigma_{CB}$	$\Delta y_{AC} \geqq -15$	$-4 \leqq \Delta y_{AB} \leqq$
C	A	$\tau_{CA} - \sigma_{CB} + \sigma_{AB}$	$\Delta y_{AC} \geqq -25$	$4 \geqq \Delta y_{CA} \geqq$

The marginal cost of transporting cargo from A to C, M_{AC} in Table 24, resu from the fact that sending an additional loaded ship from A to C reduces by one the net surplus of empty ships at A and increases by one the net surplus of empty ships at C. This reduces by one the number of monthly empty sailings from A to B and increases by one the number of monthly empty sailings from C to B, resulting in a saving of σ_{AB} and an addition to cost of σ_{CB}. These, together with the direct marginal cost τ_{AC} make up the total marginal cost given in line 5 of Table 24.

The limit of feasibility is obtained from Table 19 where it is given that to reduce cargo transportation from A to C by more than 15 ships per month would result in negative shipments.

The limits of applicability of the marginal cost coefficients are set by the fact that a surplus of only four empty ships per month is initially available at Port C. If cargo transportation from A to C is reduced by more than four ships per month C becomes a deficit port, and the lines of flow of empty shipping must change. Similarly, the upper limit of applicability of the marginal cost coefficients is a consequence of the fact that the initial net surplus of empty ships at A is six. If cargo transportation from A to C is increased by more than six ships per month, A becomes a deficit port and the lines of flow of empty shipping must be changed.

By an analogous argument we can justify line 6 of Table 24 which shows the marginal cost of cargo transportation from C to A, and the limits of feasibility and applicability of the coefficients.

The analysis employed to find the various marginal cost coefficients and their limits of applicability suggests that we can identify facets of the efficient point set by the lines of flow of empty ships. Diagrams (such as Fig. 52) indicating such lines of flow are called <u>linear graphs</u> and are analogous to diagrams of electrical networks commonly employed in physics.

<u>Four-Port Example</u>. We shall assume that the required pattern of cargo transportation (in shiploads per month) is as given in Table 25.

Table 25

Required Flows From Port...	To Port A	B	C	D
A	0	8	0	9
B	5	0	10	3
C	16	2	0	7
D	1	5	21	0

Table 26

Ports	Outflow of Loaded Ships	Inflow of Loaded Ships	Net Surplus
A	17	22	5
B	1[illegible]	15	-3
C	2[illegible]	31	6
D	[illegible]	19	-8

Our first problem is to find an efficient way to satisfy the required pattern of cargo transportation. We shall assume

$$(13.4) \qquad \sigma_{AB} + \sigma_{CD} \leq \sigma_{AD} + \sigma_{CB}$$

We then find that the lines of flow of empty ships shown in Fig. 53 represents an efficient allocation of ships.

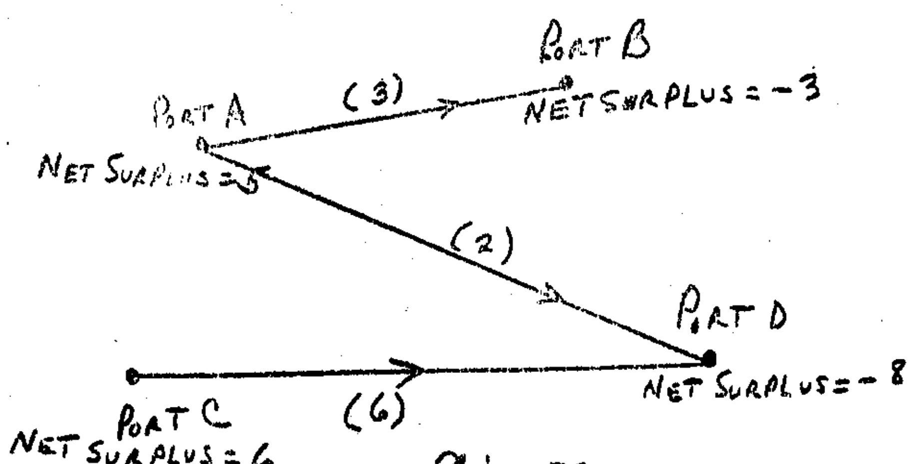


Fig. 53

Notice that now it is necessary to know and compare the times it takes to sail an empty ship from one port to another in order to determine what is efficient. If the inequality sign in (13.4) had been reversed we would have sent empty ships on the route C to B instead of A to B.

Having found an efficient point, we proceed to generate the facets of the efficient point set by finding the cost of changes in the program of cargo transportation. We need note explicitly only the marginal cost for the routes BC and CB since the remaining ones are quite analogous to those of the three-port case.

$$(13.5a) \qquad M_{BC} = \tau_{BC} + \sigma_{AB} - \sigma_{AD} + \sigma_{CD}$$

$$(13.5b) \qquad M_{CB} = \tau_{CB} - \sigma_{AB} + \sigma_{AD} - \sigma_{CD}$$

Thus, to add one shipload of cargo transported from B to C we incur the direct addition to cost of an additional loading, sailing and unloading between B and C; i.e., τ_{BC}. We also need an additional empty ship at B and we acquire an additional empty ship at C. We obtain the required empty ship at B from A, directing one which previously went to Port D, and we send the extra empty ship from C to D. The limits of feasibility is

$$(13.6) \qquad \Delta y_{BC} \geq -10, \text{ from Table 25}$$

while the limit of applicability of equations (13.5a) and (13.5b) are

$$(13.7) \qquad \begin{aligned} 2 \geq \Delta y_{BC} \geq -3 \\ 3 \geq \Delta y_{CB} \geq -2 \end{aligned}$$

The opportunity cost of making changes in the equivalence ratios between cargoes on the various routes is given by the ratio of the marginal costs involved.

In terms of the institutional setting in which the problem of allocating shipping is met, the history of the dry-cargo shipping industry affords examples of two extremely different institutional structures. During both World Wars there existed a central agency, or coordinated central agencies, which alloc[illegible]d shipping for all the Allied Powers. For such an agency, the information resultin from the analysis illustrated by our examples, is of direct operating importance. Not only would such an agency wish to route shipping efficiently for a given geographical pattern of demand for cargo in order to deal intelligently with competing demands by other agencies for the services of scarce shipping, but it must know the opportunity cost of making changes in the program.

The opposite extreme is an institutional pattern in the history of shipping is the peace-time organization of the tramp shipping industry. Prior to 1935, this industry was characterized by almost perfect competition. Apart from the effects of uncertainty regarding future demand, it can be shown that competition between ship-owners would result in an optimal solution to the routing problem. Freight rates competitively arrived at would give expression to the marginal cost coefficients we have derived.

Lecture No. 14 -- June 2, 1949.

A STATIC MULTI-PORT TRANSPORTATION MODEL

The present model employs data showing movements of dry cargo in 1925, shown in Tables 21, 22 and 23 (see pp. 106-108). These data represent shipments between areas, but for simplicity we assume that the entire traffic of an area goes through its representative port. Although the data we use were actually generated in a market situation, we shall first assume them to be given as a desired pattern of cargo transportation to a central shipping authority whose job it is to perform the indicated transportation, unchanging from year to year, at minimum cost in terms of shipping.

We shall assume that the time a ship spends on an empty voyage is proportional to the distance it sails.

Our first problem is to lay out the flows of empty movements so as to minimize the amount of shipping engaged in these movements. This amount equals the amount of ship-use committed to these movements each year, the unit of ship-use being the use of one million tons of cargo-carrying capacity for the time it takes a ship to sail 1,000 nautical miles. (We disregard the slight dependence of a ship' carrying capacity on the length of the voyage.)

The method we use is that of trial and error. First we shall draw arbitrarily lines of flow of empty movements from surplus to deficit areas, and then we shall try to make successive improvements in terms of minimizing ship-use. We begin at Yokohama, a surplus area, and decide tentatively to send empty ships to Sydney and to San Francisco. We also send empty ships from Lisbon to the two deficit ports above. This is shown in Fig. 54, where the numbers written next to each port show the net surplus of empty shipping at that port.

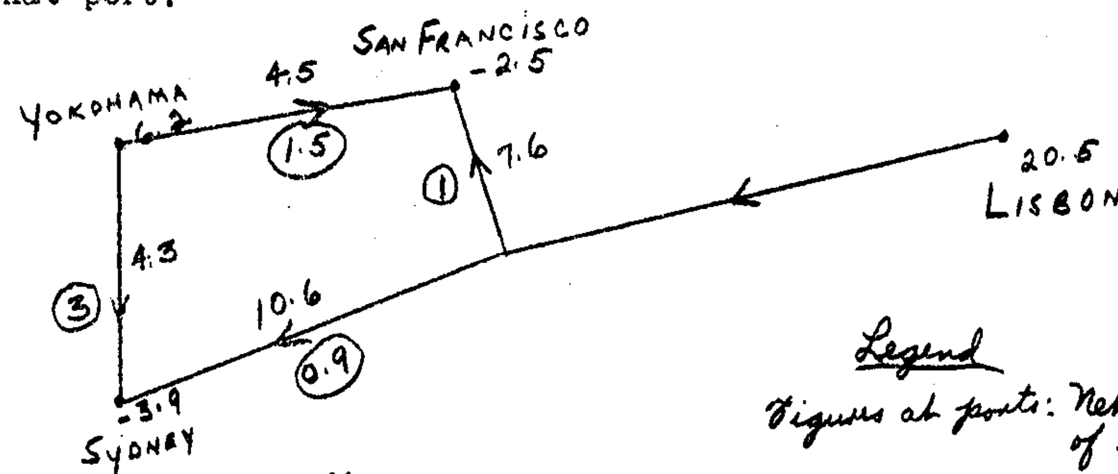


Fig 54

Legend

Figures at ports: Net Surplus of ships.

Figures with routes:
Encircled: Flows of empty ships.
Not Circled: Distance between ports.

Notice that if we start at any port, say Lisbon, and trace the routes in one direction; e.g., clockwise, we end up at Lisbon without having traced any route twice. (We trace the routes irrespectiv of their direction.) Thus the graph in Fig. 54 forms a closed figur Such a graph we shall call a circuit. In the present case we have a circuit consisting of four routes.

The question immediately arises: Can a circuit be the whole or a part of our optimal graph? To answer this question we assume that Fig. 54 is part of an over-all routing plan which is claimed to be efficient. We now tentatively make such modifications as we can in the flows of empty ships on this circuit provided that these modifications are such that they do not affect any flows of empty ships on other routes. A modification of the routing plan of this type is called a circular transformation. Our object is to see whether we can find a circular transformation which will make us better off. In order to determine this, however, we need the performance times of empty ships on these routes. From our selection of units in which to measure ship-use it follows that we take our unit of time as the time it takes for an empty ship to sail 1,000 nautical miles. Having assumed performance time to be proportional to distance sailed, we obtain the uncircled numbers attached to the lines of flow in Fig. 54.

We now consider the following circular transformation. Add one unit of shipping to the annual flow of empty ships from Yokohama to San Francisco. To balance the given net surplus at San Francisco we take one unit of empty shipping off the Lisbon-to-San Francisco route. Similarly, to balance the net surpluses at Lisbon, Sydney and Yokohama we must add one unit of shipping annually to the Lisbon to-Sydney route and subtract one unit from the Yokohama-to-Sydney route. These changes are shown in Fig. 55.

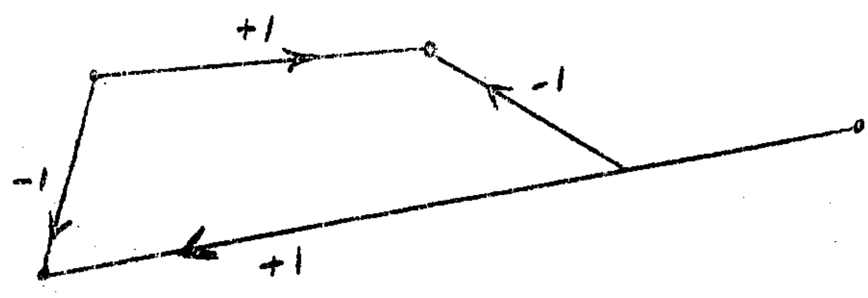


Fig. 55

This transformation is said to have a modulus of +1. If two ships were added and subtracted on the same routes, the modulus woul be +2. If one ship were added where it is now subtracted and conversely, the modulus would be -1; etc.

To determine whether a transformation of modulus +1 has left us better or worse off we go around the circuit starting, say, at Lisbon and sum the resulting changes in shipping employed. On the Lisbon-Sydney route we add 10.6 units of ship-use to our costs annually. On the Yokohama-Sydney route we decrease our costs by 4 units of ship-use. Similarly, for the other routes. The net chan in cost is given in Table 27.

Table 27

+	-
4.5	7.6
10.6	4.3
15.1	11.9

Net change = 3.2 millions of tons of cargo-carrying capacity.

It is clear from Table 27 that we would make ourselves better off if we chose a negative modulus for our transformation; i.e., if we added ships to the Lisbon-San Francisco and Yokohama-Sydney routes and decreased the empty sailings on the Lisbon-Sydney and Yokohama-San Francisco routes. We could continue to do this until empty sailings on one of the routes were reduced to zero. On which route this would occur, and by how many units we could reduce ship-use depends on the flows of empty ships specified in the original route plan. These numbers give us the highest and (algebraically) lowest limit to the modulus with which we can carry out the circula transformation.

Although in the present case we could improve our costs by not using one of the routes, it is conceivable that a circuit could exist such that the net change in cost resulting from acircular transformation would be zero. Such a circuit is called neutral (no matter how many routes it contained) and could conceivably be part of an optimal graph. To test the neutrality of a circuit it i sufficient to trace out the routes, say, in a clockwise direction, adding the empty voyage times on those routes having the same direction and subtracting the performance times on those going in the opposite direction. If this sum is zero, the circuit is neutra In general, neutral circuits occur only by accident. There is, however, one systematic cause of neutral circuits, illustrated in Fig. 56.

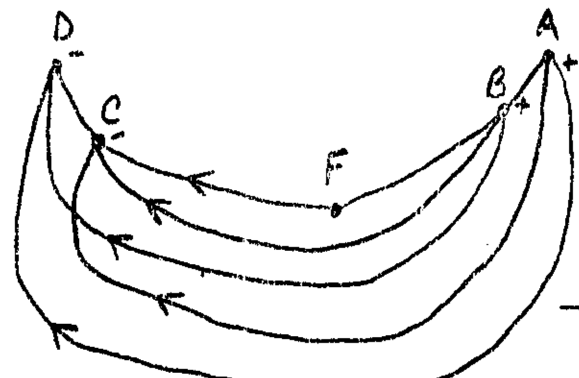


Fig 56

The ports A and B are surplus ports; the ports C and D are deficit ports. Choose a fixed point F, past which all routes must go. Denote the distances from F to each port by the small letter corresponding to the letter indicating the port. Since voyage time is assumed proportional to distance, we have

Table 28

Route	Voyage Time	Sign with which term occurs in sum
AC	a + c	+
AD	a + d	-
BC	b + c	-
BD	b + d	+

Clearly such a circuit (where all routes have to pass the same "obstacle") will always be neutral.

In order to find the most economical modulus of the circular transformation we made above we shall assume the initial flows of shipping on the various routes. These are the circled numbers attached to each line of flow in Fig. 54. Of the two routes whose traffic we diminish the Lisbon-Sydney route has the smaller initial number of units of ship-use, 0.9 units. Therefore, the most economical modulus of our transformation is -0.9. Carrying out the transformation, the routing plan now appears as in Fig. 57.

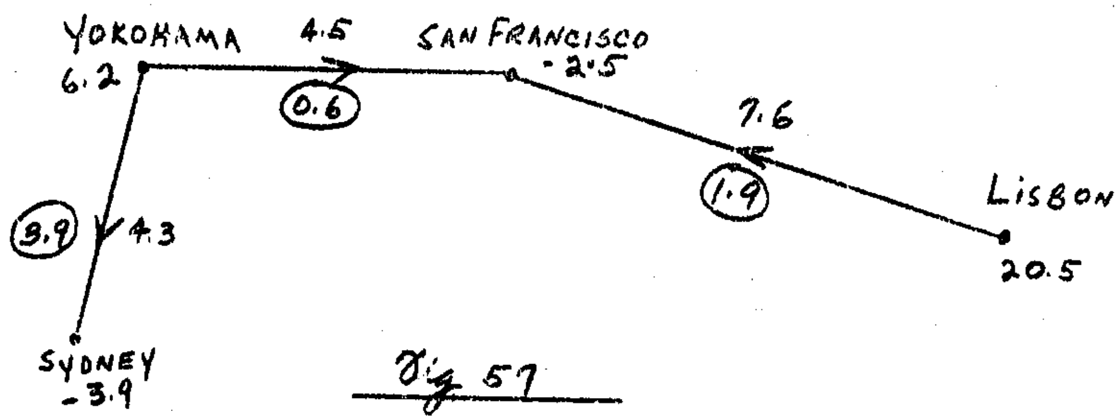


Fig 57

The route Lisbon-Sydney has been abandoned for empty ships. Using the trial and error method we continue our search for an opti route plan. The ports we consider next are given in Fig. 59. It seems clear that New York and Rotterdam are best supplied with empt shipping from Lisbon. Consider now the South American port Antofagasta. From which surplus ports should it be supplied? Conceival it could be supplied from both Yokohama and Lisbon which in additio to our previous routes gives us the graph in Fig. 59.

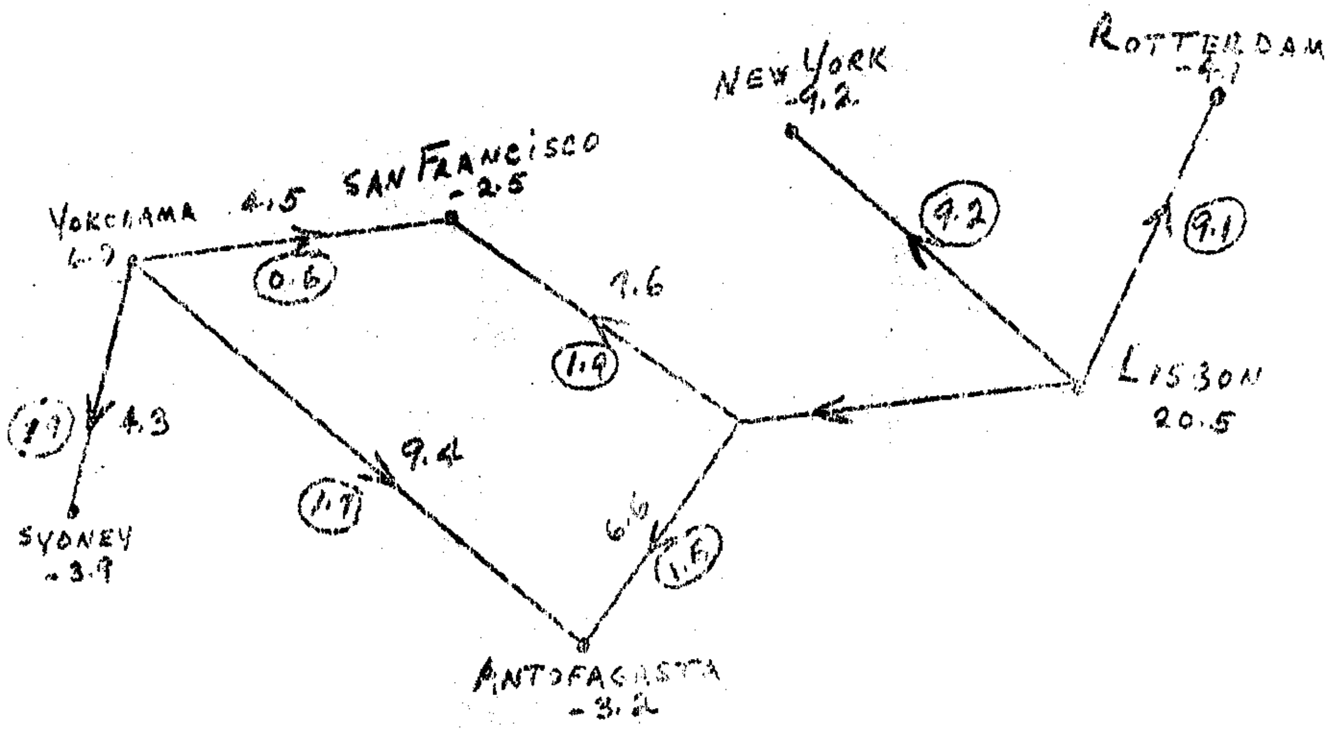


Fig 58

We see that the routes Lisbon-Antofagasta, Yokohama-Antofagasta, Yokohama-San Francisco and Lisbon-San Francisco form a four-route circuit. Tracing the routes clockwise to test for neutrality, we find

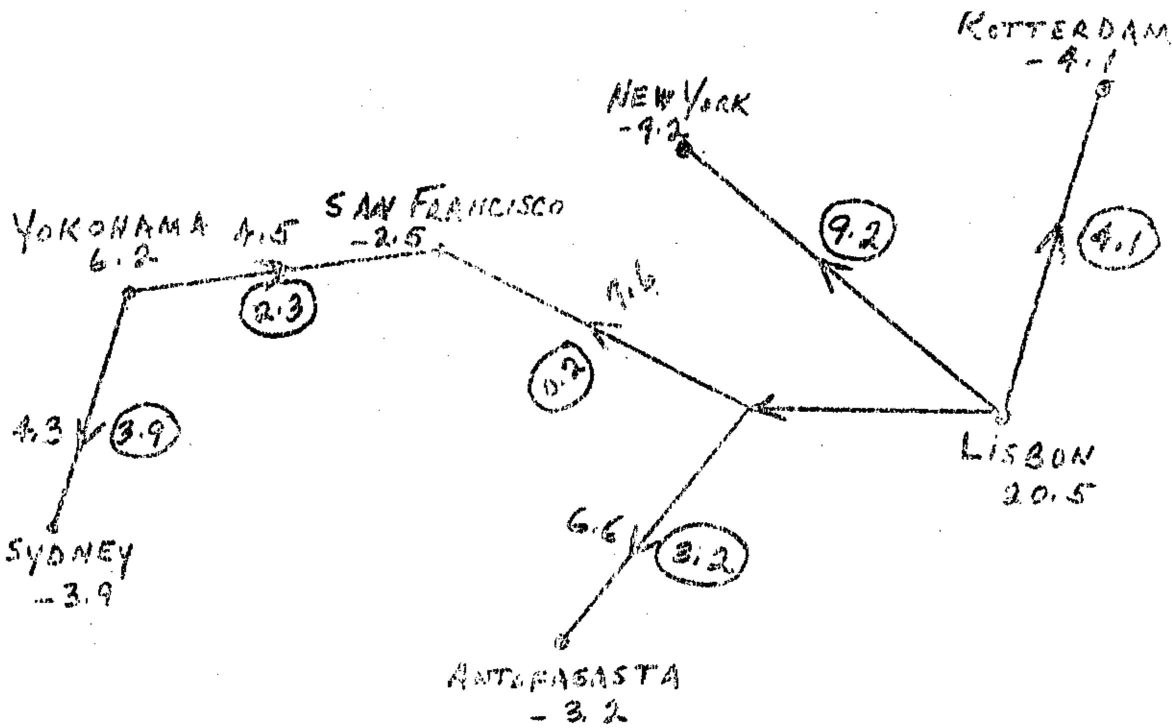
Table 29

-	+
9.4	6.6
7.6	4.5
17.0	11.1

Net change = -5.9 units of ship-use.

Thus, the circuit is not neutral and we can improve our routes by decreasing traffic on the Yokohama-Antofagasta and Lisbon-San Francisco routes. Since the smaller initial traffic is on the route Yokohama-Antofagasta, this route will "break" first, thus making the most economical modulus of our transformation 1.7. The routes now appear as in Fig. 59.

Exercise. Complete the trial and error method of constructing an optimal graph for all ports, following the rule of balancing the given net surpluses of shipping at each port. The form of the answer should be a map indicating the lines of flow of empty shipping, and optimal magnitudes of the flows. As an optional exercise try to determine whether the optimal graph is unique, and whether the optimal flows on its routes are unique.



ROTTERDAM
-4.1
NEW YORK
-9.2
9.2
4.1
YOKOHAMA
6.2
4.5
SAN FRANCISCO
-2.5
2.3
LISBON
20.5
4.3
3.9
SYDNEY
-3.9
6.6
3.2
-3.2

Fig 59

ERRATA

Change p. 121 to p. 120. Change p. 122 to p. 121.
Change "Fig. 59" on p. 120 (revised) to "Fig. 58".
Line 8 from bottom of p. 120 (revised) - Change "Fig. 60" to "Fig. 59".
Page 121 (revised) - Change "Fig. 60" to "Fig. 59".
In Table 23, p. 108, the distance "Athens to Buenos Aires" should be changed from 6.7 to 7.1; the distance "Athens to Antofagasta" should be changed from 8.1 to 8.5; and the distance "Athens to St. Thomas" should be changed from 4.7 to 5.0. Also, the distance "Athens to New York" should be changed from 4.7 to 4.

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Lecture No. 15 — June 7, 1949.

Uniqueness of Optimal Graph. We distinguish between two cases.

Case 1. There exists an optimal graph which contains a circuit. Then, as we have previously shown, this circuit must be neutral. Therefore, in such a case the solutions for optimal flows of ballast traffic on the routes of that optimal graph is not unique; for we are indifferent as between circular transformations on neutral circuits.

Case 2. There exists no optimal graph which contains a circuit. In this case we assert without proof that the optimal graph is unique; and we shall show that the flows along the lines of such a graph are also unique.

Assume that the optimal graph in Fig. 60 is a graph which contains no circuits, but connects every pair of ports considered. Such a graph is called a tree.

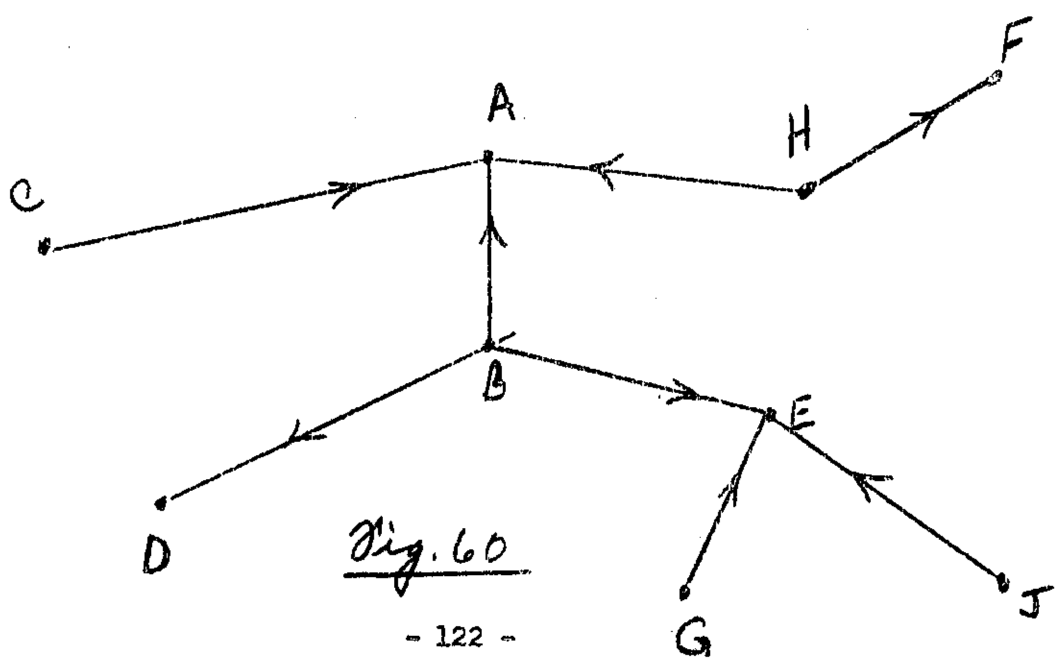

Fig. 60

It is a property of trees that if any "branch" (or route) is cut, the graph breaks into two separate parts, these parts separately being trees. (A single port to which no route leads is here considered as a special case of a tree.) In the present illustration, eliminate the route B to A. This leaves us with two trees, one consisting of the routes between B, D, E and G and the other of the routes between A, C, H and F. We then compute the joint net surplus (deficit) of ships of all ports touched by one of the trees, say CAHF. Since our optimal graph was given to be unique by assertion, there is only one way in which the net deficit of this tree can be supplied; i.e., by the route B to A, and this net deficit is a unique number.

The same argument applies, of course, to other routes and to route plans which consist initially of several disconnected trees. In the latter case we simply treat each tree separately.

We can briefly suggest the mathematical reason for the uniqueness of the optimal graph, if no optimal graph contains a circuit. Our rule for finding an optimum graph has been that no circular transformation made possible through the addition of one route to the presumed optimal graph should decrease cost. A graph so found is unique if a "local" optimum is always an absolute optimum—"local" to be understood with reference to a space X in which the coordinates are the flows x_{ij} of ballast traffic. Mathematically, an optimum solution is defined by requiring that

$$(15.1) \qquad F(x) = \sum_{i,j} C_{ij}^{-} x_{ij}; \text{ to be minimized subject to}$$

$$(15.2) \qquad \sum_{j} x_{ij} - \sum_{j} x_{ji} = a_i; \text{ and}$$

$$(15.3) \qquad x_{ij} \geq 0.$$

The points $x = [x_{12}, x_{13}, \ldots, x_{n-1\,n}]$ (flow patterns of empty shipping) which satisfy (15.2) and (15.3) constitute a convex point set in the space X: (15.2) and (15.3) restrict the x's to a linear sub-space in which all coordinates are positive. Therefore, if $x^{(1)}$ and $x^{(2)}$ are points satisfying (15.2) and (15.3) then

$$(15.4) \qquad x = \lambda x^{(1)} + (1-\lambda)x^{(2)}; \text{ where } 0 \leq \lambda \leq 1$$

is also such a point. Equation (15.4) defines convexity. Thus, we are minimizing a linear function on a convex set.

Now, suppose a local minimum $F[x^{(2)}]$ were reached at a point $x^{(2)}$, and a lower absolute minimum $F[x^{(1)}]$ at a point $x^{(1)}$:

$$F[x^{(1)}] < F[x^{(2)}]$$

Then $F(x)$ becomes by (15.4) a linear function of λ, and by taking λ small enough we can bring x arbitrarily close to $x^{(2)}$ and still have

$$F(x) < F[x^{(2)}].$$

But this contradicts the assumption of a local minimum different from the absolute minimum.

Lecture No. 16 -- June 9, 1949.

Several methods have been suggested by members of the class for finding the optimal graph by direct procedures not involving construction of tentative circuits. While these methods cannot be regarded as necessarily leading to the optimal graph, they are quite useful in that they may provide a tentative graph very close to the optimum, and sometimes give the optimal graph in one step (although, without giving certainty that the graph so obtained is actually optimal).

THE LOCATIONAL POTENTIAL

Having arrived at an optimum outlay of flows of ballester on an optimal graph; i.e., at an efficient point, x, our problem is now to find exchange ratios between the following commodities:

(a) Final -- Cargo transportation services on all routes.
(b) Intermediate -- Appearances of ships in various locations.
(c) Primary -- Use of shipping.

We shall use Table 18, p. 96, reproduced below and now to be considered as part of a larger technology matrix involving many ports and all routes between them.

Table 18

Prices	Commodities	Transp. Cargo A to B	Empty Sailing A to B	Transp. Cargo B to A	Empty Sailing B to A
		$\bar{x}_{AB}$	x_{AB}	$\bar{x}_{BA}$	x_{BA}
p_{AB}	y_{AB}	1			
p_{BA}	y_{BA}			1	
p_A	y_A	-1	-1	1	1
p_B	y_B	1	1	-1	-1
q	z	$-\tau_{AB}$	$-\sigma_{AB}$	$-\tau_{BA}$	$-\sigma_{BA}$

Note that this larger matrix, in the columns referring to the routes AB and BA, has zero coefficients in all rows except those appearing in Table 18. (In other columns referring to other routes involving either A or B, it does have additional non-vanishing coefficients in the last three rows appearing in Table 18, but not in the first two rows shown.)

We rename y_A and y_B as denoting the <u>rate of appearances of empty ships</u> at A and B, respectively, no matter from what activity the ship came.

Suppose we are given that the ballast traffic route A to B is part of our optimal graph. That is, on the facet corresponding to our optimal graph,

$$(16.1) \qquad x_{AB} > 0$$

Then, applying the zero-profit rule to this activity, we have

$$(16.2) \qquad -p_A + p_B - \sigma_{AB} q = 0$$

If we normalize on the "price" of the use of shipping by $q = 1$, we express the "prices" of all other commodities in ship-use-equivalents. This leads to

$$(16.3) \qquad p_B = p_A + \sigma_{AB}$$

Equation (16.3) says that the value of the appearances of one ship per month in Port B exceeds the value of the appearance of one ship per month in Port A by the cost of sending one empty ship per month from A to B. This equation reflects the circumstance that we found it economical regularly to send empty shipping from A to B in our optimal routing plan. Further, such an equation must hold for all pairs of ports connected by empty shipping routes in our optimal graph. The interconnected "prices" p_A, p_B, ...so obtained can be regarded as a set of differential valuations on the location of empty shipping. Such a set of valuations we shall call a <u>potential function</u> of the location of a ship.

Assume that our optimal graph does not consist of two or more disconnected parts and that therefore in our optimal graph empty shipping routes penetrate to all ports. Then we can find the potential function in a way which we shall describe in terms of the example of 1925 dry cargo movements.

In the first place, since we define our valuations as relating to a rate of appearance of one ship per month, we must choose the month as the unit of time, rather than (as previously) the time required to sail 1,000 nautical miles. We shall assume that the latter time is 1/5 months. This means that the figures in Table 23 (p.108) need to be divided by 5 in order to yield sailing time in months.

Now, we place an arbitrary value zero on the appearance of an empty ship per month in an arbitrarily selected port, say Athens.

$$p_{Ath.} = 0$$

If we then consider the route Athens-to-Durban, the value of an appearance per month in Durban is defined to be

$$p_{Durb} = p_{Ath} + \sigma_{Ath\text{-}Durb} = 0 + 1.04 = 1.04$$

according to equation (16.3), where $\sigma_{Ath\text{-}Durb}$ is now measured in months.

Repeating this procedure for other ports to which ballast traffic moves from Athens yields the prices (values of the potential function) for Sydney (1.80), Bombay (0.74), Singapore (1.14), Odessa (0.14) and Lagos (0.92) as shown in Fig. 61.

Potential Function
of Cargo Shipping
According to 1925 Program

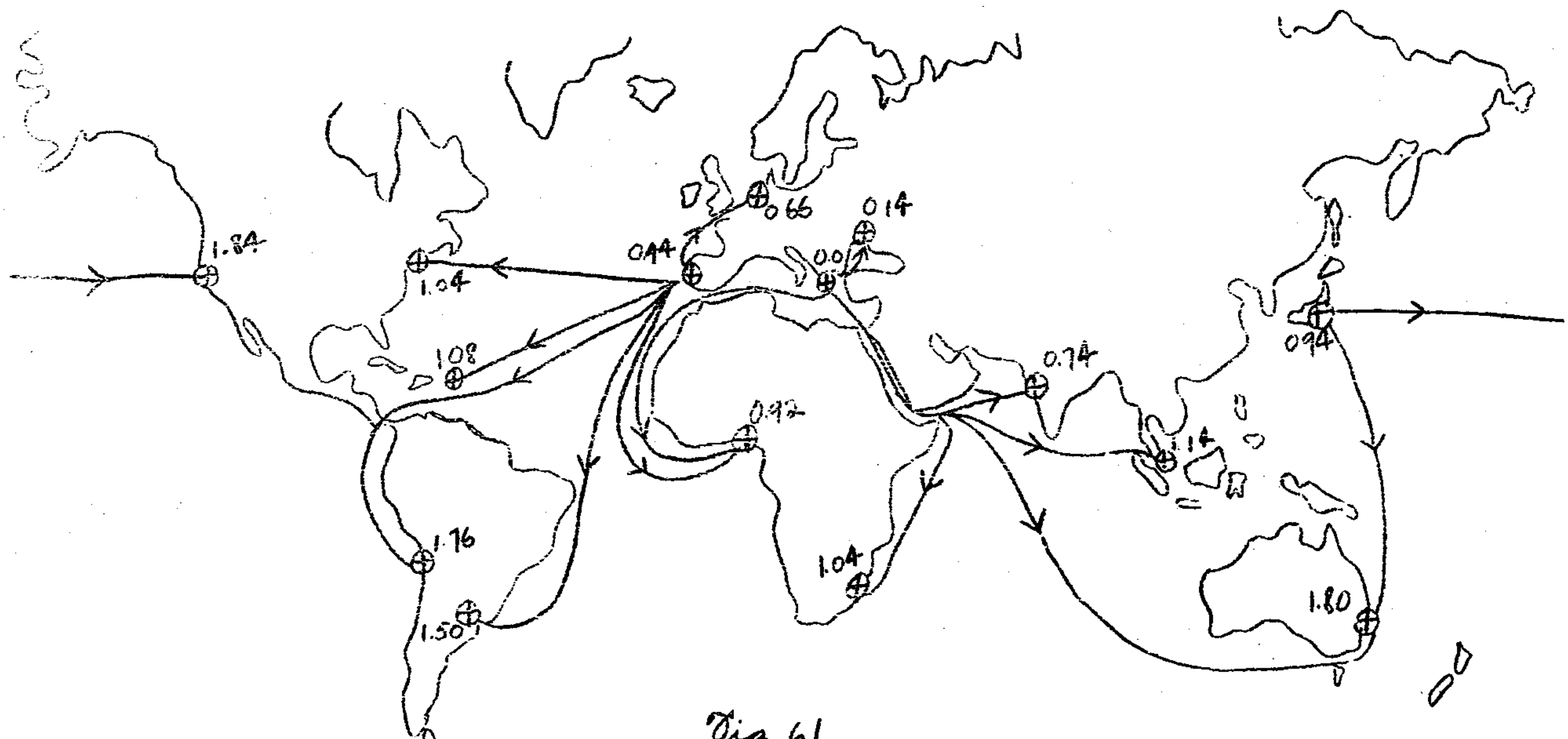

Fig 61

Turning now to Sydney, where the value of the appearance of one ship per month was found to be 1.80, we move counter to the direction of the route along the Yokohama-Sydney route. Thus, we must subtract from the value at Sydney the sailing time 0.86 from Yokohama to Sydney to obtain the value of an appearance per month at Yokohama; namely: 0.94. Having obtained this figure, to find the value of an appearance per month at San Francisco we add the sailing time 0.90 for the route Yokohama to San Francisco to the value of an appearance at Yokohama, giving us the valuation 1.84 of one appearance per month at San Francisco.

Turning now to Lagos, we trace the Lisbon-Lagos route to obtain the value of a ship per month at Lisbon and by reapplying the same procedure at all the remaining ports which are served by Lisbon only. Notice that we have moved only along empty shipping routes actually in our graph.

Since as we noted in a previous lecture, an optimal graph may be identified with a facet of the efficient point set, and since if the optimal graph were to change our potential function would change, the potential function applies only within a facet. If we were to move to another facet we would have to find the new values of the potential function corresponding to that facet.

Potential and Marginal Cost. Take any two ports in which the potential function is defined, say San Francisco and Antofagasta. We now inquire what is the marginal cost of sending an additional cargo per month from San Francisco to Antofagasta. Applying the zero-profit rule on the cargo transportation activity on that route (and having already normalized on q) we have

$$(16.4) \quad p_{Fran-Ant} - p_{Fran} + p_{Ant} - \tau_{Fran-Ant} = 0$$

or

$$(16.5) \quad p_{Fran-Ant} = \tau_{Fran-Ant} + p_{Fran} - p_{Ant}$$

Thus, the marginal cost of an additional monthly cargo from Fran to Ant consists of the direct cost ($\tau_{Fran-Ant}$) plus the loss in potential ($p_{Fran} - p_{Ant}$) sustained by a ship as a result of carrying cargo on that route. The loss in potential reflects all the repercussions of the change, thus giving as a simple way to compute the indirect costs of a change in the shipping program. To see this, follow the repercussions of the change indicated in the flows of empty shipping on each route of the chain Antofagasta-Lisbon-Lagos-Athens-Sydney-Yokohama-San Francisco, and convince yourself that the net indirect cost is obtained by the same calculation that has entered into the definition of the potential difference between San Francisco and Antofagasta. Notice that the calculations serve only for changes in our shipping program small enough to keep us on the same facet; i.e., small enough so that on no route in our original optimal graph ballast traffic would have to become negative to balance the given net surpluses.

Uses of the Potential Function. Several examples follow in which knowledge of the potential function would prove helpful.

(1) In a war situation where shipping is centrally directed and choices must be made as between obtaining raw material from one part of the world or another, indirect as well as direct costs must be considered. Indeed in such cases the indirect costs may often be the determining factor in the decision taken.

Lecture No. 16 (Cont'd).

(2) Knowledge of the potential function is useful to a central shipping agency in its efforts to decide between competing claims for the use of the same shipping. This function conveys the information necessary to determine the opportunity cost of meeting one claim, in terms of the size of the other claim which has to be denied.

(3) If a ship is half-loaded when a convoy is ready to sail, should the ship be sent or should it be loaded fully and held to wait for the next convoy? This is left as an exercise.

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Lecture No. 17 — June 14, 1949.

DYNAMIC MODELS

In our discussion of the optimal allocation of shipping we have considered only static models. However, some of our concepts carry over to dynamic models. For example, if changes over time in the shipping program are small enough so that the same graph is optimal after as well as before the change, then the potential function remains a useful concept. In other respects, however, dynamic models require modification and extension of our previously developed methods. We can date transportation services, and regard flows of cargo in different periods as different commodities to which we apply the same efficiency analysis. This analysis would be appropriate if the changing program is fully known at the beginning of the period covered by it. However, an interesting problem of short notice arises if a change in program occurs during the period, such that the central shipping authority has insufficient time to make optimal adjustments. This will typically involve additional costs. Such costs we call the cost of short notice. As an extreme example, an empty ship may be half way from Lisbon to New York when a sudden change in program requires that the ship be recalled to Lisbon. A route of this kind; i.e., Lisbon to mid-Atlantic to Lisbon, cannot be part of an optimal graph for a static program.

Shipping Under Competition with Static Demand. While both tramp and liner shipping exist in the peace-time shipping industry, we shall assume that only tramp shipping exists. We also assume conditions of unchanging demand. These assumptions amount to assuming perfect competition in a static world. We shall attempt to show that under these conditions our theory is useful in explaining the formation of freight rates. For purposes of simplicity we shall assume that ship-owners lease their ships to operators-entrepreneurs on a time-charter basis; i.e., the ship-owner provides a ship, crew, supplies, insurance, repairs, etc., and receives payment on the basis of the time the ship is under the direction of the operator.

Classification of Costs to Operator:

I. Costs Proportional to Time Spent in Any Activity

A. Time Charter Rate, to cover
1. Wages
2. Supplies
3. Insurance
4. General Repairs
5. Depreciation
6. Profit to Ship-owner (quasi-rent)

The cost items IA, 1 to 6, appear to the operator as a single cost, the time-charter rate.

II. Costs Proportional to Sea Time.

A. Fuel for Propulsion of Ship.

III. Costs Depending on Cargo and Route.

A. Port and Canal Charges
B. Repair Needs Specific to Route
C. Cost of Cargo Handling

Under the conditions assumed, and with every ship operator maximizing his profit, the freight rate for one shipload of a given cargo on a given route equals:

costs specific to cargo and route + (sea time x fuel cost per day) + (cargo transportation time x daily time charter rate) + net loss in potential in money units sustained by a ship performing this transportation;

where

cargo transportation time = time for loading, sailing and discharging + time allowance for fueling and making standard repairs.

potential difference in money units = potential difference in time units x time charter rate per day + daily fuel cost + costs specific to route.

potential difference in time units = sailing time + time allowance for fueling and making standard repairs.

Freight rates between any two ports will then be such that, no matter what combination of routes the ship operator chooses to make up a round voyage, he will come out with zero profit, provided he does not let his ship sail empty on routes that do not belong to the optimal graph of ballast traffic. In particular on such routes, freight rates must be such that the ship operator is indifferent as to whether he should send the ship along an empty route or to accept cargo.

Actually, the tramp shipping industry operates in a highly dynamic and uncertain world, not in the static one we have assumed, and therefore incurs certain costs arising from uncertainty. Thus, our conclusion that optimal shipping will result from a competitive market may or may not apply to dynamic situations where uncertainty exists.

<u>Railroad Transportation</u>. Railroad transportation differs from shipping in several important respects. Fixed equipment, as distinguished from rolling equipment, constitutes a much higher proportion of railroad capital than does the comparable class of assets for the shipping industry. Because of the nature of fixed equipment in the railroad industry, indivisibilities are quite important.

Marginal cost pricing of railroad transportation will not, therefore, recover the total investment except at very high densities of traffic. It has been argued, however, by Hotelling that marginal cost pricing constitutes an optimal set of freight rates. This would require that either the capital be compensated by subsidy or that there be government ownership with deficits financed from general taxes. Carrying Hotelling's reasoning to its conclusion, we would have to have freight rates which depended on the direction of movement. Thus, in the U.S., the rates on east to west traffic would generally be lower than the rates on west to east traffic since empty freight cars move largely from east to west.

In practice, the ICC takes the position that total cost should be recoverec from freight (and passenger) charges. The problem then arises of selecting those commodities to be charged higher than marginal cost prices. This selectic is usually made on the basis of the elasticity of demand for railroad transportation. There is, however, no clear principle of selection discernible in the literature.

The difficulties of formulating a rate situation for railroads based on marginal cost are considerable in that such rates would have to depend on the composition of traffic. This implies that, in the absence of a market with contracts for forward performance, the composition of traffic must be forecastec by the railroad system. If the composition of traffic fluctuates, the problem of forecasting is, of course, complicated. For seasonal fluctuations the difficulty is not extreme. Good seasonal rate patterns could be found and thes would give proper incentive with regard to the location of storage of commoditi

Cyclical fluctuations present a more serious problem. Here it is difficul to say whether reforming the freight rate structure would, if possible, be of net benefit or harm to the whole economy. E.g., low freight rates in depressio and high rates in booms might through their effects on movements in the general price level and the anticipations created thereby result in cyclical fluctuatio of greater amplitude.

Thus, it may be that a freight rate structure based on marginal cost, and therefore providing proper incentives with regard to location of industry, could only be defended in a situation in which over-all cyclical fluctuations had already been minimized.

END