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Homogeneous Systems in Mathematical Economics: A Comment

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Your comment is appreciated.
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Summary

In a recent issue of this journal, Professor Tintner has developed some important general theorems giving conditions under which maximizing behavior leads to relations which are homogeneous of degree zero in certain variables.

The purpose of the present note is to derive these theorems in a more elementary way in which, further, Professor Tintner's assumptions of differentiability are shown to be unnecessary.

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1. Introduction:

In a recent paper[1] in this journal, Professor Tintner has developed some highly useful general theorems bringing together the various problems in economic theory which lead to relations homogeneous of degree zero in certain variables. These theorems were derived by Professor Tintner under the assumption that the various functions involved are differentiable twice[2]. Now, there appears to be a rapidly developing tendency in modern mathematical economics to avoid assumptions of differentiability which seem to have no economic relevance and which frequently are unnecessary. Indeed, the use of methods of the differential

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1. G. Tintner, "Homogeneous Systems in Mathematical Economics," Econometrica, Vol. 16, October, 1948, pp. 273-294.

2. Tintner, op. cit., p. 277.

calculus occasionally leads to unnecessarily complicated derivations and expositions[3]. It is the purpose of the present note to show that Professor Tintner's chief results can be obtained without any assumptions of differentiability and in a simpler manner. It will consequently be noted that some remarks of Professor Tintner implying that differentiability is a necessary condition for homogeneity are misleading.

The tools used by Professor Tintner are the following two theorems. Let the behavior of some individual be described by saying that he maximizes $g(x, x^*, x^{**})$ with respect to $\underline{x}$ and $\underline{x}^*$ subject to the restrictions, $h^{(k)}(x, x^*, x^{**}) = 0$ $(k=1, 2, \ldots, N)$. Here $\underline{x}$, $\underline{x}^*$, $\underline{x}^{**}$ are vectors; let $\underline{x}_1, \ldots, \underline{x}_p$ be the components of $\underline{x}$.

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3. This point has been especially stressed by Samuelson. See P. A. Samuelson, Foundations of Economic Analysis, Cambridge, Massachusetts, 1947, pp. 70-76, 107-112; "Comparative Statics and the Logic of Economic Maximizing," Review of Economic Studies, Vol. XIV (1), 1946-'47, pp. 41-43. There is a growing body of economic literature dealing with cases in which the individual is confronted with choice among alternatives which are convex combinations of a finite number of basic alternatives; in such cases, the standard methods of the calculus are useless. See P. A. Samuelson, "Comparative Statics," op. cit.; J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton, 1944; J. von Neumann, "Über ein Ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," Ergebnisse eines Mathematischen Kolloquiums, Vol. 8 (1935-1936), pp. , translated as, "A Model of General Economic Equilibrium," Review of Economic Studies, Vol. XIII (1), 1945-'46, pp. 1-9; T. C. Koopmans, "Optimum Utilization of the Transportation System," (abstract), Econometrica, Vol. 16, January, 1948, pp. 66-68; "A Mathematical Model of Production," (abstract), Econometrica, Vol. 17, January, 1949, pp.

It is true, however, as pointed out by one of the reviewers of Samuelson's book, that the differentiability or non-differentiability of the utility or revenue function or of the constraints on behavior is of importance in characterizing the behavior of the decision variables with respect to variations in the initial conditions. In the non-differentiable case, the decision variables may well be constant for small variations in initial conditions, with a discontinuous jump in response to larger variations.

See K. E. Boulding, "Samuelson's Foundations: The Role of Mathematics in Economics," Journal of Political Economy, Vol. LVI, June, 1949, p. 194.

Theorem 5: if the funtion g which is to be maximized and the side conditions $h^{(k)}$ are either independent of the variables that are components of x^{**} or homogeneous of some arbitrary degrees in the same variables, then the solutions $x_1, \ldots, x_p$ are homogeneous of zero degree in these variables.

Theorem 6: If the function g which is to be maximized and the side conditions $h^{(k)}$ are either independent of the variables that are components of x^* and x^{**} or homogeneous of some arbitrary degrees in the same variables, then the solutions $x_1, x_2, \ldots, x_p$ are homogeneous of zero degree in the variables that are components of x^{**}.

It is to be noted that, strictly speaking, in the conclusion of these theorems the word, "homogeneous," should be replaced by the phrase, "positive homogeneous." A function $f(u, u^*)$ of two vectors u, u^* is said to be positive homogeneous of degree K in u if $f(tu, u^*) = t^K f(u, u^*)$ for every positive value of t. Positive homogeneity is a weeker condition than homogeneity; it is the relevant condition in economics since, for example, utility is no longer maximized by consuming the same quantities if prices and income are all made negative but with the same magnitudes. The distinction between positive homogeneity and ordinary homogeneity is of importance in what follows. It may also be observed that we need only assume positive homogeneity in f and $g^{(k)}$ in the relevant variables. From now on, it will be assumed that the word, "homogeneous," has been replaced by the phrase, "positive homogeneous," wherever it occurs in Theorems 5 and 6.

2. Derivation of Theorem 5 and 6

Theorem 5 will first be derived from Theorem 6.

Proof of Theorem 5: Let $x^1 = (x, x*) = (x_1, \ldots, x_r)$, and let x'' be a vector without components. Under the assumption of Theorem 5, f and $g^{(k)}$ are each positive homogeneous of some degree in x'', $x**$, or independent of them. Then, Theorem 6 applies, with x^1, x'' replacing x, $x**$ respectively, so that the solutions $x_1, \ldots, x_r$, and, in particular $x_1, \ldots, x_p$, are positive homogeneous of degree zero in $x**$.

Proof of Theorem 6: Let $\hat{x}$, $\hat{x}*$ maximize f subject to $g^{(k)} = 0$ for a given $x**$. Then, by dfinition of maximum,

(1) $$f(\hat{x}, \hat{x}*, x**) \geqq f(x, x*, x**),$$

for all x, $x*$ such that,

(2) $$g^{(k)}(x, x*, x**) = 0,$$

where, also,

(3) $$g^{(k)}(\hat{x}, \hat{x}*, x**) = 0.$$

Take any $t > 0$. Assume f position homogeneous of degree K_o in $x*$, $x**$, $g^{(k)}$ positive homogeneous of degree K_k in $x*$, $x**$. Multiply through both sides of (1) by $t^{K_o} > 0$. By the definition of positive homogeneity,

$$f(\hat{x}, t\hat{x}*, tx**) = t^{K_o} f(\hat{x}, \hat{x}*, x**),$$
$$f(x, tx*, tx**) = t^{K_o} f(x, x*, x**),$$

so that

(4) $$f(x, tx*, tx**) \geqq f(x\ tx*, tx**),$$

for all x, $x*$ satisfying (2). Multiply through in (2) by t^{Kk}; by the definition of positive homogeneity,

$$g^{(k)}(x, tx*, tx**) = t^{Kk} g^{(k)}(x, x*, x**),$$

so that (2) is equivalent to,

$$g^{(k)}(x,\ tx^*,\ tx^{**}) = 0. \tag{5}$$

Therefore, (4) holds for all $\underline{x}$, $\underline{x}^*$ satisfying (5). Replace $\underline{x}^*$ by $(\underline{x}^*/\underline{t})$ in (4) and (5); this is permissible since $t \neq 0$.

$$f(\hat{x},\ t\hat{x}^*,\ tx^{**}) \geqq f(x,\ x^*,\ tx^{**}), \tag{6}$$

for all $\underline{x}$, $\underline{x}^*$ satisfying,

$$g^{(k)}(x,\ x^*,\ tx^{**}) = 0. \tag{7}$$

Finally, multiply through in (3) by $\underline{t}^{K_k}$. By the same argument as before,

$$g^{(k)}(\hat{x},\ t\hat{x}^*,\ tx^{**}) = 0. \tag{8}$$

From (6), (7) and (8) and the definition of a constrained maximum, $\hat{\underline{x}}^*$, $t\hat{x}^*$ maximize $\underline{f}(\underline{x}, \underline{x}^*, \underline{tx}^{**})$ subject to $\underline{g}^{(k)}(\underline{x}, x^*, \underline{tx}^{**})=0$. Since $\hat{\underline{x}}$ is unchanged by multiplying all the components of x^{**} by a positive constant, the solutions $x_1, \ldots, x_p$ are positive homogeneous at degree zero in x^{**}.

It has also been proved, incidentally, that the components at the solution $\underline{x}^*$ are positive homogeneous of degree one in $\underline{x}^{**}$. Thus, in Professor Tintner's discussion of monopoly[4] it can be shown that the prices charged by a monopolist are homogeneous of degree one in the prices charged on atomistic markets.

3. Kinked Demand Curve in Oligopoly.

Since differentiability has been shown to be irrelevant for the validity of the basic theorems on homogeneous functions, it cannot be, as Professor Tintner states in several places, that lack of differen-

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4. Op. cit., pp. 284-5.

tiability has been shown to be irrelevant for the validity of the basic theorems on homogeneous functions, it cannot be, as Professor Tintner states in several places, that lack of differentiability explains lack of homogeneity. In the case of the kinked demand curve in oligopoly[5], the point seems to be that the established price must be taken as a variable distinct from price as a decision variable; since it is a datum, it is to be included, along with the prices on atomistic markets, in the components of $\underline{x}^{**}$. The analyses of oligopoly then proceeds as before, so that output is positive homogeneous of degree zero in all the component of $\underline{x}^{**}$; but of course, output need not be homogeneous of degree zero in the prices on atomistic markets alone.

4. Lumpiness.

Similarly, it cannot be that the existence of lumpiness disturbs homogeneity because of the lack of differentiability. Mathematically, lumpiness may be represented by letting certain variables assume only integer values. The previous definitions of homogeneity do not have any meaning in this case. A more restricted definition must be introduced.

Definition: A function $\underline{f}(\underline{u}, \underline{u}^*)$ is said to be positive integral homogeneous of degree $\underline{K}$ in $\underline{u}$ if $f(tu, u^*) = t^K f(u, u^*)$, for every positive integer $\underline{t}$.

It can easily be seen that both of the theorems previously stated are valid if the words, "positive homogeneous" are everywhere replaced by the words, "positive integral homogeneous." The proofs are identical.

5. Op. cit., p. 287

Of course, the conclusion that a certain variable is positive integral homogenous of degree zero in certain other variables says less than that that variable is positive homogenous of degree zero in those variables, and the difference may be important for certain purposes.