

Computational Suggestions for Maximizing a  
Linear Function Subject to Linear Inequalities

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Let the function to be maximized be

$$(1) \quad y = \pi'_* \cdot x$$

(y scalar,  $\pi_*$  and x vectors), subject to linear inequalities

$$(2) \quad \phi_k + \pi'_k \cdot x \geq 0 \quad k = 1, \dots, K, \quad \phi_k \text{ scalar, } \pi_k \text{ vector.}$$

We distinguish two main cases (A and B).

A. When one possesses an initial point  $x_0$  satisfying (2).

A.1 Traversal Method (Koopmans' Suggestion).

Find the largest value  $\theta_0$  of the scalar  $\theta$  such that

$$(3) \quad x = x_0 + \theta \pi_*$$

satisfies (2) and write

$$(4) \quad x_1 = x_0 + \theta_0 \pi_*$$

Then for at least one value,  $k_1$  say, of k

$$(5) \quad \phi_{k_1} + \pi'_{k_1} \cdot x_1 = 0.$$

If it is true for only one value of k, determine  $\lambda_1, \mu_1$ , <sup>scalars</sup> such that

$$(6) \quad \pi' \cdot (\lambda_1 \pi_{k_1} + \mu_1 \pi_*) = 0.$$

This is impossible only if  $\pi_* = c \pi_{k_1}$ , c scalar, in which case  $x_1$

already maximizes y. Determine  $\theta_1$  as the largest value of  $\theta$  for

which

$$(7) \quad x = x_1 + \theta(\lambda_1 n_{k_1} + \mu_1 n_*)$$

satisfies (2) and write

$$(8) \quad x_2 = x_1 + \frac{1}{2}\theta_1(\lambda_1 n_{k_1} + \mu_1 n_*).$$

Proceed with  $x_2$  as previously with  $x_0$ . If at the  $n$ -th step more than one  $k_n$  satisfies an equation like (5), select one arbitrarily, or use an average of all  $n_{k_n}$  that satisfy (5). (The latter procedure depends on the normalization used for the vector  $n_k$ .)

#### A.2 Plane Intersection Method (G. Brown's Suggestion).

Having obtained (4), intersect the plane

$$(9) \quad x = x_1 + \lambda n_{k_1} + \mu n_*$$

( $\lambda$  and  $\mu$  freely variable scalar parameters) successively with each of the hyperplanes

$$(10) \quad \phi_k + n_k' \cdot x = 0.$$

The intersections consist of  $K$  straight lines inside (9). The segments of these lines on which (2) is satisfied form a convex polygon. On that polygon select a point on which  $y$  reaches its maximum. Generally there is just one such point, for which write  $x_2$ . Now there are two variants.

A.2a Plane determined by normal to the convex set (2). If  $x_2$  is unique there are at least two values of  $k$  for which (10) is satisfied. Take any one of these, or take their average, as  $n_{k_2}$  and proceed as in (9) with 1 replaced by 2.

A.2b Plane determined by normal within boundary of convex set (2).

Having arrived close to the maximum, it may be desirable to attempt not to lose any of the equalities (10) once they are satisfied. Let  $x_n$  be such that

$$(11) \quad \phi_k + \pi'_k \cdot x_n = 0 \quad \text{for } k = k_1, k_2, \dots, k_{r_n}.$$

In the space of the vectors  $\psi$  such that

$$(12) \quad \psi' \cdot \pi_{k_r} = 0, \quad r = 1, \dots, r_n$$

choose the vector of steepest ascent, i.e., the vector satisfying

$$(13) \quad \psi' \cdot \psi = 0 \quad \psi' \cdot \pi_* = \text{maximum},$$

and use that vector as  $\pi_{k_n}$  in (9). This will have been wasted effort if the resulting  $x_{n+1}$  fails to satisfy (11). In conversation, Von Neumann made objections to this variant if the number of dimensions is large. It would seem that each step becomes computationally more expensive as  $r_n$  grows.

B. How to obtain an initial point satisfying (2).

B.1 Selective Penetration Method (G. Brown's Suggestion).

Take an arbitrary initial point  $x_0$ . This point partitions the set S of inequalities (2) into two subsets  $S_0$  and  $S'_0$ , those of  $S_0$  being satisfied by  $x_0$ , those of  $S'_0$  not being satisfied by  $x_0$ . If  $S'_0$  is empty, the goal has been achieved. If it is not, select arbitrarily an inequality of  $S'_0$ , numbered  $k_0$ , say. Use  $\pi_{k_0}$  as the vector  $\pi_*$  in (1) indicating the "desired direction," use the inequalities of  $S_0$  instead of the full set of conditions (2) and apply method A

until a point is reached in which the inequality numbered  $k_0$  is satisfied. Call that point  $x_1$  and proceed with a new partition  $S_1$ . Obviously

$$S_0 \in S_1 \in S_2 \dots$$

The method can fail only if no internal point exists.

B.2 Guided Penetration Method (Koopmans' Suggestion)

Instead of selecting an arbitrary  $\pi_k$  of  $S'_0$  to be the  $\pi_*$  in (1), take

$$\pi_* = \sum_{k \in S'_0} (\phi_k + \pi'_k \cdot x) \frac{n_k}{\pi'_k \cdot n_k}$$

This is the vector sum of the normals dropped from  $x_0$  onto the planes

$$(\phi_k + \pi'_k \cdot x) = 0 \quad k \in S'$$

B.2a Keep the  $\pi_*$  so selected constant while making a number of iterative improvements to  $x_0$  by method A, always requiring that the inequalities  $S_0$  be preserved.

B.2b Be willing to sacrifice some inequalities of  $S_0$  if thereby a larger number of inequalities of  $S'_0$  can be satisfied. In this case one determines  $\phi_0$  in (4) in such a way as to minimize the number of inequalities in  $S_1$ .

For neither of these variants certain attainment of the objective (if attainable) has been proved. They might, however, work faster than the selection method. The second alternative is suspect if the convex (2) is not bounded.

C. General Observations

The general idea underlying the foregoing method is to make big jumps rather than "crawling along the edges" of the convex (2). All

methods indicated are based on some idea of steepest ascent, and thus depend on the units of measurement of the variables  $x$ .

The determination of the "normal"  $n_{k_1}$  in (5)--by any of the methods suggested--is likely to present practical difficulties in the neighborhood of the maximum. It may then be useful to take the vector sum of normals of unit length to all planes  $n_k \cdot x' = 0$  to which  $x_0$  has a distance less than a small amount  $\xi$ .