Limit Theorems for Functionals of Sums that Converge to Fractional Brownian and Stable Motions

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Abstract. Consider $S_k = \sum_{j=1}^{k} X_j$, where $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, $k \geq 1$, with ξ_j, $-\infty < j < \infty$, iid belonging to the domain attraction of a strictly stable law with index $0 < \alpha \leq 2$. Under certain conditions on c_j, it is known that for $\gamma_n = n^H \tau_n$, $0 < H < 1$, with τ_n slowly varying, $\gamma_n^{-1} S_{[nt]}$ converges in distribution to a fractional stable motion. In addition, if $f(y)$ is such that $\int \left(|f(y)| + |f(y)|^2\right) dy < \infty$, then for β_n such that $\beta_n \to \infty$ and $\frac{\beta_n}{n} \to 0$ (in particular $\beta_n = \gamma_n$), $\frac{\beta_n}{n} \sum_{k=1}^{n} f\left(\frac{\beta_n}{\gamma_n} S_k\right)$ converges in distribution to $L_1^0 \int_{-\infty}^{\infty} f(y)\, dy$, where L_1^0 is the local time of the fractional stable motion. In this paper we obtain further results, motivated by asymptotic inference.

We obtain the convergence in distribution for $\frac{\beta_n}{n} \sum_{k=1}^{n} h\left(\frac{\beta_n}{\gamma_n} S_k, \frac{\beta_n}{\gamma_n} S_{k+r}\right)$, $r \geq 1$, as well as for $\frac{\beta_n}{n} \sum_{k=1}^{n} f\left(\frac{\beta_n}{\gamma_n} S_k\right) \sigma(\omega_k)$ and $\frac{\beta_n}{n} \sum_{k=1}^{n} h\left(\frac{\beta_n}{\gamma_n} S_k, \frac{\beta_n}{\gamma_n} S_{k+r}\right) \sigma(\omega_k) \sigma(\omega_{k+r})$, $r \geq 1$, for suitable $f(x)$ and $h(x, y)$ and for suitable $\sigma(\omega_k)$, where $\omega_k = \sum_{j=-\infty}^{k} d_{k-j} \eta_j$ such that $(\xi_j, \eta_j), -\infty < j < \infty$, are iid with $E[\eta_1^2] < \infty$ but possibly with $E[\eta_1] \neq 0$. For $h(x, y)$, the limits are different for the cases $\beta_n = \gamma_n$, $\frac{\beta_n}{\gamma_n} \to 0$ and $\frac{\beta_n}{\gamma_n} \to \infty$.

If in addition $\int_{-\infty}^{\infty} f(y)\, dy = 0$, then when $1/3 < H < 1$ (which cannot be relaxed), $\sqrt{\frac{\gamma_n}{n}} \sum_{k=1}^{n} f(S_k)$ converges in distribution. Similarly but when possibly $\int_{-\infty}^{\infty} f(y)\, dy \neq 0$, the same is true for $\sqrt{\frac{\gamma_n}{n}} \sum_{k=1}^{n} f(S_k) \omega_k$, where ω_k is as before but with $E[\eta_1] = 0$.

All the above convergencies are also shown to hold jointly.

JEL Classification: C22, C23

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