Limit Theorems for Functionals of Sums that Converge to Fractional Stable Motions

P.Jeganathan

Indian Statistical Institute, Bangalore Centre, Bangalore 560059, India.

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Abstract. Consider $X_k = \sum_{j=0}^{\infty} c_j \xi_{k-j}$, $k \geq 1$, where c_j are constants and ξ_j are iid random variables belonging to the domain attraction of a strictly stable law with index $0 < \alpha \leq 2$. Let $S_k = \sum_{j=1}^{k} X_j$. Under certain conditions on c_j, it is known that for a suitable slowly varying function $\kappa_1(n)$ and for a suitable constant $0 < H < 1$, $\left(n^H \kappa_1(n)\right)^{-1} S_{[nt]} \Longrightarrow$ to a fractional stable motion (indexed by H). In addition, it is known that if $f(y)$ is such that $\int \left(|f(y)| + |f(y)|^2\right) dy < \infty$, then $n^{-(1-H)} \kappa_1(n) \sum_{k=1}^{n} f(S_k) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f(y)\, dy$, where L_t^x is the local time of the fractional stable motion at x upto time t.

In this paper we obtain three further results, motivated by asymptotic inference for certain nonlinear time series models. First, we show that if in addition $\int_{-\infty}^{\infty} f(y)\, dy = 0$, then when $1/3 < H < 1$ (which probably cannot be relaxed), $\sqrt{n^{-(1-H)} \kappa_1(n)} \sum_{k=1}^{n} f(S_k) \Longrightarrow W\sqrt{bL_1^0}$, where W is standard normal, independent of L_1^0, and b is a constant having an explicit expression in terms of the distributions of S_k, $k \geq 1$. (A continuous time version of this result holds also.)

Now let, for $\nu \geq 1$, $\omega_k = \sum_{j=k-\nu+1}^{k} d_{k-j} \eta_j$ where (ξ_j, η_j), $-\infty < j < \infty$, are iid with ξ_j as before and $E[\eta_1] = 0$, $E[\eta_1^2] < \infty$ and $E[|\eta_1 \xi_1|] < \infty$. Then if $1/3 < H < 1$ as above but possibly $\int_{-\infty}^{\infty} f(y)\, dy \neq 0$, we show that $\sqrt{n^{-(1-H)} \kappa_1(n)} \sum_{k=1}^{n} f(S_k)\omega_k \Longrightarrow W\sqrt{b^* L_1^0}$. The constant b^* in the limit will be similar to that of b in the first result.

It is further shown that $n^{-(1-H)} \kappa_1(n) \sum_{k=1}^{n} f(S_k, S_{k+1}, ..., S_{k+r}) \Longrightarrow L_1^0 \int_{-\infty}^{\infty} f_*(x)\, dx$ for all $0 < H < 1$ and for all suitable $f(x_0, ..., x_r)$, $r \geq 1$, where $f_*(x) = E[f(x, x + S_1, ..., x + S_r)]$.

These convergencies are also shown to hold jointly with certain other random quantities.

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